International Journal of Mathematics and Physical Sciences Research ISSN 2348-5736 (Online)
Vol. 12, Issue 2, pp: (25-31), Month: October 2024 - March 2025, Available at: www.researchpublish.com

Fractional Derivatives of Four Types of Matrix Fractional Functions

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DOI: https://doi.org/10.5281/zenodo.14199158

Published Date: 21-November-2024

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of four types of matrix fractional functions by using some methods. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. Moreover, our results are generalizations of ordinary calculus results.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, matrix fractional functions.**

I. INTRODUCTION

Fractional calculus originated in 1695 and almost at the same time as traditional calculus. Fractional calculus is considered to be a useful tool for understanding and simulating many natural and artificial phenomena. It has developed rapidly in different scientific fields in the past few decades, including not only mathematics and physics, but also engineering, biology, economics and chemistry [1-13].

However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [14-18]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we use some techniques to study the following fractional differential problem of four types of matrix fractional functions:

$$\left({}_0D_x^{\alpha}\right)^m\left[sinh_{\alpha}\left(\rho cos_{\alpha}(sAx^{\alpha})\right)\otimes_{\alpha} cos_{\alpha}\left(\rho sin_{\alpha}(sAx^{\alpha})\right)\right],$$

$$\left({}_0D_x^{\alpha}\right)^m\left[cosh_{\alpha}\left(\rho cos_{\alpha}(sAx^{\alpha})\right)\otimes_{\alpha} sin_{\alpha}\left(\rho sin_{\alpha}(sAx^{\alpha})\right)\right],$$

$$\left({}_0D_x^{\alpha}\right)^m\left[cosh_{\alpha}\left(\rho cos_{\alpha}(sAx^{\alpha})\right)\otimes_{\alpha} cos_{\alpha}\left(\rho sin_{\alpha}(sAx^{\alpha})\right)\right],$$

$$\left({}_0D_x^{\alpha}\right)^m\left[sinh_{\alpha}\left(\rho cos_{\alpha}(sAx^{\alpha})\right)\otimes_{\alpha} sin_{\alpha}\left(\rho sin_{\alpha}(sAx^{\alpha})\right)\right],$$

where $0 < \alpha \leq 1$, ρ, s are real numbers, m is any positive integer, and A is a real matrix. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this paper. In addition, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper.

Definition 2.1 ([19])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^{\alpha}\right)^m[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([20])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([21])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([22])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{5}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([23])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\left(g_\alpha(x^\alpha)\right) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha n}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\left(f_\alpha(x^\alpha)\right) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n}. \tag{11}$$

Definition 2.6 ([24])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$.

Definition 2.7: If the complex number $z = p + iq$, where p, q are real numbers, and $i = \sqrt{-1}$. p, the real part of z, is denoted by $\text{Re}(z)$; q the imaginary part of z, is denoted by $\text{Im}(z)$.

Definition 2.8 ([25])**:** If $0 < \alpha \leq 1$, t is a real number, and A is a matrix. The matrix α-fractional exponential function, matrix α-fractional cosine function, and matrix α-fractional sine function are defined as follows:

$$E_\alpha(tAx^\alpha) = \sum_{n=0}^{\infty}(tA)^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}, \quad (12)$$

$$cos_\alpha(tAx^\alpha) = \sum_{n=0}^{\infty}(tA)^{2n} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \quad (13)$$

$$sin_\alpha(tAx^\alpha) = \sum_{n=0}^{\infty}(tA)^{2n+1} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \quad (14)$$

In addition, the matrix α-fractional hyperbolic cosine function and matrix α-fractional hyperbolic sine function are defined as follows:

$$cosh_\alpha(tAx^\alpha) = \frac{1}{2}[E_\alpha(tAx^\alpha) + E_\alpha(-tAx^\alpha)] = \sum_{n=0}^{\infty}(tA)^{2n}\frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \quad (15)$$

and

$$sinh_\alpha(tAx^\alpha) = \frac{1}{2}[E_\alpha(tAx^\alpha) - E_\alpha(-tAx^\alpha)] = \sum_{n=0}^{\infty}(tA)^{2n+1}\frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \quad (16)$$

Theorem 2.9 (matrix fractional Euler's formula)([26])**:** *If* $0 < \alpha \leq 1$, *and* A *is a real matrix, then*

$$E_\alpha(iAx^\alpha) = cos_\alpha(Ax^\alpha) + isin_\alpha(Ax^\alpha). \quad (17)$$

Theorem 2.10 (matrix fractional DeMoivre's formula)([27])**:** *If* $0 < \alpha \leq 1$, p *is an integer, and* A *is a real matrix, then*

$$[cos_\alpha(Ax^\alpha) + isin_\alpha(Ax^\alpha)]^{\otimes_\alpha p} = cos_\alpha(pAx^\alpha) + isin_\alpha(pAx^\alpha). \quad (18)$$

Definition 2.11: The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS

In this section, we use some methods to obtain arbitrary order fractional derivative of four types of matrix fractional functions. At first, two lemmas are needed.

Lemma 3.1: *If* $0 < \alpha \leq 1$, ρ, s *are real numbers, and* A *is a real matrix, then*

$$sinh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)$$
$$= sinh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha cos_\alpha\left(\rho sin_\alpha(sAx^\alpha)\right) + i\cdot cosh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha sin_\alpha\left(\rho sin_\alpha(sAx^\alpha)\right). \quad (19)$$

$$cosh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)$$
$$= cosh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha cos_\alpha\left(\rho sin_\alpha(sAx^\alpha)\right) + i\cdot sinh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha sin_\alpha\left(\rho sin_\alpha(sAx^\alpha)\right). \quad (20)$$

Proof By matrix fractional Euler's formula and matrix fractional DeMoivre's formula,

$$sinh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)$$
$$= sinh_\alpha\left(\rho cos_\alpha(sAx^\alpha) + i\cdot\rho sin_\alpha(sAx^\alpha)\right)$$
$$= sinh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha cosh_\alpha\left(i\rho sin_\alpha(sAx^\alpha)\right) + cosh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha sinh_\alpha\left(i\rho sin_\alpha(sAx^\alpha)\right)$$

$= sinh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha cos_\alpha(\rho sin_\alpha(sAx^\alpha)) + i\cdot cosh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha sin_\alpha(\rho sin_\alpha(sAx^\alpha))$.

And

$cosh_\alpha(\rho E_\alpha(isAx^\alpha))$

$= cosh_\alpha(\rho cos_\alpha(sAx^\alpha) + i\cdot \rho sin_\alpha(sAx^\alpha))$

$= cosh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha cosh_\alpha(i\rho sin_\alpha(sAx^\alpha)) + sinh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha sinh_\alpha(i\rho sin_\alpha(sAx^\alpha))$

$= cosh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha cos_\alpha(\rho sin_\alpha(sAx^\alpha)) + i\cdot sinh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha sin_\alpha(\rho sin_\alpha(sAx^\alpha))$. q.e.d.

Lemma 3.2: *If* $0 < \alpha \leq 1, \rho, s$ *are real numbers, and* A *is a real matrix, then*

$$sinh_\alpha(\rho E_\alpha(isAx^\alpha)) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\rho^{2n+1}[cos_\alpha((2n+1)sAx^\alpha) + i\cdot sin_\alpha((2n+1)sAx^\alpha)]\,, \quad (21)$$

$$cosh_\alpha(\rho E_\alpha(isAx^\alpha)) = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\rho^{2n}[cos_\alpha(2nsAx^\alpha) + i\cdot sin_\alpha(2nsAx^\alpha)]\,. \quad (22)$$

Proof $sinh_\alpha(\rho E_\alpha(isAx^\alpha))$

$= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(\rho E_\alpha(isAx^\alpha))^{\otimes_\alpha (2n+1)}$

$= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\rho^{2n+1}E_\alpha(i(2n+1)sAx^\alpha)$ (by matrix fractional DeMoivre's formula)

$= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\rho^{2n+1}[cos_\alpha((2n+1)sAx^\alpha) + i\cdot sin_\alpha((2n+1)sAx^\alpha)]$.

And

$cosh_\alpha(\rho E_\alpha(isAx^\alpha))$

$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}(\rho E_\alpha(isAx^\alpha))^{\otimes_\alpha 2n}$

$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\rho^{2n}E_\alpha(i2nsAx^\alpha)$ (by matrix fractional DeMoivre's formula)

$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\rho^{2n}[cos_\alpha(2nsAx^\alpha) + i\cdot sin_\alpha(2nsAx^\alpha)]$. q.e.d.

Theorem 3.3: *If* $0 < \alpha \leq 1$, ρ, s *are real numbers,* m *is any positive integer,* A *is a real matrix, and* E *is the unit matrix, then*

$$\left({}_0D_x^\alpha\right)^m[sinh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha cos_\alpha(\rho sin_\alpha(sAx^\alpha))]$$
$$= (sA)^m\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\rho^{2n+1}(2n+1)^m cos_\alpha\left((2n+1)sAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right), \quad (23)$$

$$\left({}_0D_x^\alpha\right)^m[cosh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha sin_\alpha(\rho sin_\alpha(sAx^\alpha))]$$
$$= (sA)^m\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\rho^{2n+1}(2n+1)^m sin_\alpha\left((2n+1)sAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right), \quad (24)$$

$$\left({}_0D_x^\alpha\right)^m[cosh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha cos_\alpha(\rho sin_\alpha(sAx^\alpha))]$$
$$= (sA)^m\sum_{n=0}^{\infty}\frac{1}{(2n)!}\rho^{2n}(2n)^m cos_\alpha\left(2nsAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right), \quad (25)$$

$$\left({}_0D_x^\alpha\right)^m[sinh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha sin_\alpha(\rho sin_\alpha(sAx^\alpha))]$$
$$= (sA)^m\sum_{n=0}^{\infty}\frac{1}{(2n)!}\rho^{2n}(2n)^m sin_\alpha\left(2nsAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right). \quad (26)$$

Proof By Lemma 3.1 and Lemma 3.2,

$\left({}_0D_x^\alpha\right)^m[sinh_\alpha(\rho cos_\alpha(sAx^\alpha))\otimes_\alpha cos_\alpha(\rho sin_\alpha(sAx^\alpha))]$

$$= \left({}_0D_x^\alpha\right)^m\left[\mathrm{Re}\left\{sinh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right\}\right]$$

$$= \mathrm{Re}\left\{\left({}_0D_x^\alpha\right)^m\left[sinh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right]\right\}$$

$$= \mathrm{Re}\left\{\left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^\infty \frac{1}{(2n+1)!}\rho^{2n+1}[cos_\alpha((2n+1)sAx^\alpha) + i\cdot sin_\alpha((2n+1)sAx^\alpha)]\right]\right\}$$

$$= \sum_{n=0}^\infty \frac{1}{(2n+1)!}\rho^{2n+1}(2n+1)^m(sA)^m cos_\alpha\left((2n+1)sAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right)$$

$$= (sA)^m\sum_{n=0}^\infty \frac{1}{(2n+1)!}\rho^{2n+1}(2n+1)^m cos_\alpha\left((2n+1)sAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right).$$

$$\left({}_0D_x^\alpha\right)^m\left[cosh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha sin_\alpha\left(\rho sin_\alpha(sAx^\alpha)\right)\right]$$

$$= \left({}_0D_x^\alpha\right)^m\left[\mathrm{Im}\left\{sinh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right\}\right]$$

$$= \mathrm{Im}\left\{\left({}_0D_x^\alpha\right)^m\left[sinh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right]\right\}$$

$$= \mathrm{Im}\left\{\left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^\infty \frac{1}{(2n+1)!}\rho^{2n+1}[cos_\alpha((2n+1)sAx^\alpha) + i\cdot sin_\alpha((2n+1)sAx^\alpha)]\right]\right\}$$

$$= \sum_{n=0}^\infty \frac{1}{(2n+1)!}\rho^{2n+1}(2n+1)^m(sA)^m sin_\alpha\left((2n+1)sAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right)$$

$$= (sA)^m\sum_{n=0}^\infty \frac{1}{(2n+1)!}\rho^{2n+1}(2n+1)^m sin_\alpha\left((2n+1)sAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right).$$

$$\left({}_0D_x^\alpha\right)^m\left[cosh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha cos_\alpha\left(\rho sin_\alpha(sAx^\alpha)\right)\right]$$

$$= \left({}_0D_x^\alpha\right)^m\left[\mathrm{Re}\left\{cosh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right\}\right]$$

$$= \mathrm{Re}\left\{\left({}_0D_x^\alpha\right)^m\left[cosh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right]\right\}$$

$$= \mathrm{Re}\left\{\left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^\infty \frac{1}{(2n)!}\rho^{2n}[cos_\alpha(2nsAx^\alpha) + i\cdot sin_\alpha(2nsAx^\alpha)]\right]\right\}$$

$$= \sum_{n=0}^\infty \frac{1}{(2n)!}\rho^{2n}(2n)^m(sA)^m cos_\alpha\left(2nsAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right)$$

$$= (sA)^m\sum_{n=0}^\infty \frac{1}{(2n)!}\rho^{2n}(2n)^m cos_\alpha\left(2nsAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right).$$

Finally,

$$\left({}_0D_x^\alpha\right)^m\left[sinh_\alpha\left(\rho cos_\alpha(sAx^\alpha)\right)\otimes_\alpha sin_\alpha\left(\rho sin_\alpha(sAx^\alpha)\right)\right]$$

$$= \left({}_0D_x^\alpha\right)^m\left[\mathrm{Im}\left\{cosh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right\}\right]$$

$$= \mathrm{Im}\left\{\left({}_0D_x^\alpha\right)^m\left[cosh_\alpha\left(\rho E_\alpha(isAx^\alpha)\right)\right]\right\}$$

$$= \mathrm{Im}\left\{\left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^\infty \frac{1}{(2n)!}\rho^{2n}[cos_\alpha(2nsAx^\alpha) + i\cdot sin_\alpha(2nsAx^\alpha)]\right]\right\}$$

$$= \sum_{n=0}^\infty \frac{1}{(2n)!}\rho^{2n}(2n)^m(sA)^m sin_\alpha\left(2nsAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right)$$

$$= (sA)^m\sum_{n=0}^\infty \frac{1}{(2n)!}\rho^{2n}(2n)^m sin_\alpha\left(2nsAx^\alpha + m\cdot\frac{T_\alpha}{4}E\right).$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we find arbitrary order fractional derivative of four types of matrix fractional functions by using some methods. In fact, our results are generalizations of traditional calculus results. In the future, we will continue to use Jumarie type of R-L fractional derivative and the new multiplication of fractional analytic functions to study the problems in fractional differential equations and applied mathematics.

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